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# Orthogonal and symplectic parabolic bundles

Indranil Biswas *, Souradeep Majumder, Michael Lennox Wong

*School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India*

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ABSTRACT

We investigate orthogonal and symplectic bundles with parabolic structure, over a curve.



## 1. Introduction

Let $X$ be an irreducible smooth complex projective curve, and let $S \subset X$ be a fixed finite subset. The notion of parabolic vector bundles on $X$ with $S$ as the parabolic divisor was introduced by Seshadri [1]. Let $G$ be any complex reductive group. The generalization of parabolic bundles to the context of $G$-bundles was carried out in [2].

Here we take $G$ to be the orthogonal or symplectic group. In these cases the parabolic bundles can be considered as a usual parabolic vector bundles with a symmetric or alternating form taking values in a parabolic line bundle; the form has to be nondegenerate in a suitable sense.

We define algebraic connection on orthogonal and symplectic parabolic bundles, and give a criterion for the existence of such a connection (see Theorems 4.1 and 4.2). This criterion is similar to the one of Weil and Atiyah [3,4] for the existence of an algebraic connection on a vector bundle over $X$.

It turns out that an orthogonal or symplectic parabolic bundle is semistable (respectively, polystable) if and only if the underlying parabolic vector bundle is semistable (respectively, polystable); see Propositions 5.6 and 5.7 and Corollary 6.2.

We also prove the following theorem (see Theorem 6.1):

**Theorem 1.1.** *Let $(E_*, \varphi)$ be an orthogonal or symplectic parabolic bundle. Then $(E_*, \varphi)$ admits an Einstein–Hermitian connection if and only if it is polystable.*

## 2. Orthogonal and symplectic structures

### *2.1. Parabolic vector bundles*

Let $X$ be an irreducible smooth complex projective curve. Fix distinct points of $X$

$$S := \{x_1, \ldots, x_n\} \subset X. \tag{2.1}$$

* Corresponding author.
*E-mail addresses:* indranil@math.tifr.res.in (I. Biswas), souradip@math.tifr.res.in (S. Majumder), wong@math.tifr.res.in (M.L. Wong).

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Let $E \longrightarrow X$ be a vector bundle. A *quasi-parabolic* structure on $E$ over $S$ is a filtration of subspaces

$$E_{x_i} =: F_{i,1} \supsetneq \cdots \supsetneq F_{i,j} \supsetneq \cdots \supsetneq F_{i,a_i} \supsetneq F_{i,a_i+1} = 0 \tag{2.2}$$

over each point of $S$. A *parabolic* structure on $E$ is a quasi-parabolic structure as above together with real numbers

$$0 \leq \alpha_{i,1} < \cdots < \alpha_{i,j} < \cdots < \alpha_{i,a_i} < 1 \tag{2.3}$$

associated to the quasi-parabolic flags. (See [1], [5, p. 67], [6].) The numbers in (2.3) are called *parabolic weights*.

A vector bundle equipped with a quasi-parabolic structure over $S$ and parabolic weights as above is called a *parabolic vector bundle* with parabolic structure over $S$. The subset $S$ is called the *parabolic divisor* for the parabolic vector bundle.

We fix the divisor $S$ once and for all. Henceforth, the parabolic divisor for all parabolic vector bundles will be $D$.

For notational convenience, a parabolic vector bundle $(E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ as above will also be denoted by $E_*$.

The *parabolic degree* is defined to be

$$\text{par-deg}(E_*) := \text{degree}(E) + \sum_{i=1}^{n} \sum_{j=1}^{a_i} \alpha_{i,j} \cdot \dim(F_{i,j}/F_{i,j+1}). \tag{2.4}$$

The real number $\text{par-deg}(E_*)/\text{rank}(E_*)$ is called the *parabolic slope* of $E_*$, and it is denoted by $\mu_{\text{par}}(E_*)$.

### 2.2. Parabolic dual and parabolic tensor product

In [6], an equivalent definition of parabolic vector bundles was given. This definition of [6] is very useful to work with; we will recall it now. Take a parabolic vector bundle $(E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ defined as in (2.2) and (2.3). For a point $x_i \in S$ and $t \in [0, 1]$, let

$$E^{i,t} \subset E$$

be the coherent subsheaf defined as follows: if $t \leq \alpha_{i,1}$, then

$$E^{i,t} = E,$$

if $t > \alpha_{i,1}$, then $E^{i,t}$ is defined by the short exact sequence

$$0 \longrightarrow E^{i,t} \longrightarrow E \longrightarrow E/F_{i,j+1} \longrightarrow 0,$$

where $j \in [1, a_i]$ is the largest number such that $\alpha_{i,j} < t$. Since $F_{i,a_i+1} = 0$ (see (2.2)), it follows that $E^{i,t} = E \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(-x_i)$ for $t > \alpha_{i,a_i}$. For $t \in [0, 1]$, define

$$E^{(t)} = \bigcap_{i=1}^{n} E^{i,t} \subset E.$$

Now we have a filtration of coherent sheaves $\{E_t\}_{t\in\mathbb{R}}$ defined by

$$E_t := E^{(t-[t])} \otimes \mathcal{O}_X(-[t]S), \tag{2.5}$$

where $[t]$ is the integral part of $t$ (so $0 \leq t - [t] < 1$). Note that

(1) the sheaf $E_t$ decreases as $t$ increases,
(2) the filtration $\{E_t\}_{t\in\mathbb{R}}$ is left-continuous, more precisely, there is an $\epsilon > 0$ such that $E_{t-\epsilon} = E_t$ for all $t$, and
(3) $E_{t+1} = E_t \bigotimes \mathcal{O}_X(-S)$ for all $t$.

From the construction of the filtration $\{E_t\}_{t\in\mathbb{R}}$ it is evident that the parabolic vector bundle $(E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ can be recovered from it. Conversely, given any filtration of coherent sheaves satisfying the above three conditions, we get a parabolic vector bundle on $X$ with parabolic structure over $S$. In [6], a parabolic vector bundle on $X$ with parabolic structure over $S$ is defined to be a filtration of coherent sheaves satisfying the above three conditions.

Let

$$\iota : X \setminus S \hookrightarrow X \tag{2.6}$$

be the inclusion of the complement. For any coherent sheaf $V$ on $X \setminus S$, the direct image $\iota_* V$ is a quasi-coherent sheaf on $X$.

Let $\{V_t\}_{t\in\mathbb{R}}$ and $\{W_t\}_{t\in\mathbb{R}}$ be the filtrations corresponding to two parabolic vector bundles $V_*$ and $W_*$ respectively. Consider the torsionfree quasi-coherent sheaf $\iota_*\left(\left(V_0 \bigotimes W_0\right)|_{X\setminus S}\right)$ on $X$, where $\iota$ is defined in (2.6). Note that $V_s \bigotimes W_t$ is a coherent subsheaf of it for all $s$ and $t$. For any $t \in \mathbb{R}$, let

$$\mathcal{E}_t \subset \iota_*((V_0 \otimes W_0)|_{X\setminus S})$$

be the quasi-coherent subsheaf generated by all $V_\alpha \bigotimes W_{t-\alpha},\ \alpha \in \mathbb{R}$. It is easy to see that $\mathcal{E}_t$ is a coherent sheaf, and the collection $\{\mathcal{E}_t\}_{t\in\mathbb{R}}$ satisfies all the three conditions needed to define a parabolic vector bundle on $X$ with parabolic structure over $S$.

The parabolic vector bundle defined by $\{\mathcal{E}_t\}_{t\in\mathbb{R}}$ is denoted by $V_* \bigotimes W_*$, and it is called the *tensor product* of $V_*$ and $W_*$.

Now consider the torsionfree quasi-coherent sheaf $\iota_* \left( (V_0^* \bigotimes W_0) \,|_{X\setminus S} \right)$ on $X$. For any $t \in \mathbb{R}$, let

$$\mathcal{F}_t \subset \iota_*((V_0^* \otimes W_0)|_{X\setminus S})$$

be the quasi-coherent subsheaf generated by all $V_\alpha^* \bigotimes W_{\alpha+t}$, $\alpha \in \mathbb{R}$. This $\mathcal{F}_t$ is a coherent sheaf, and the collection $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ satisfies the three conditions needed to define a parabolic vector bundle with parabolic structure over $S$.

The parabolic vector bundle defined by $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is denoted by $\mathcal{H}om(V_*, W_*)$.

Let $L_*^0$ be the trivial line bundle $\mathcal{O}_X$ with trivial parabolic structure (there is no nonzero parabolic weight). Note that the sheaf for $t \in \mathbb{R}$ corresponding to this parabolic line bundle is $\mathcal{O}_X([-t]D)$, so the filtration is $\{\mathcal{O}_X([-t]D)\}_{t\in\mathbb{R}}$. The *parabolic dual* of $V_*$ is defined to be

$$V_*^* := \mathcal{H}om(V_*, L_*^0).$$

If $\{U_t\}_{t\in\mathbb{R}}$ is the filtration corresponding to the parabolic vector bundle $V_*^*$, then $U_t = (V_{\epsilon-t-1})^*$, where $\epsilon$ is a sufficiently small positive real number.

We also note that $\mathcal{H}om(V_*, W_*) = W_* \bigotimes V_*^*$. (See [7,8].)

### *2.3. Orthogonal and symplectic parabolic bundles*

Fix a parabolic line bundle $L_*$.

Let $E_*$ be a parabolic vector bundle, and let

$$\varphi : E_* \otimes E_* \longrightarrow L_* \tag{2.7}$$

be a homomorphism of parabolic bundles. Tensoring both sides of the above homomorphism with the parabolic dual $E_*^*$ we get a homomorphism

$$\varphi \otimes \mathrm{Id} : E_* \otimes E_* \otimes E_*^* \longrightarrow L_* \otimes E_*^*. \tag{2.8}$$

We note that the sheaf of sections of the vector bundle underlying $E_* \otimes E_*^*$ is the sheaf of endomorphisms of $E$ preserving the quasi-parabolic flags. Sending any locally defined function $h$ to the locally defined endomorphism of $E$ given by pointwise multiplication with $h$, the trivial line bundle $\mathcal{O}_X$ equipped with the trivial parabolic structure (meaning there is no nonzero parabolic weight) is realized as a parabolic subbundle of $E_* \otimes E_*^*$. In fact, this line subbundle $\mathcal{O}_X \subset E_* \otimes E_*^*$ is a direct summand of the subbundle of $E_* \otimes E_*^*$ defined by the sheaf of parabolic endomorphisms of trace zero. Let

$$\widetilde{\varphi} : E_* \longrightarrow L_* \otimes E_*^* \tag{2.9}$$

be the homomorphism defined by the composition

$$E_* = E_* \otimes \mathcal{O}_X \hookrightarrow E_* \otimes (E_* \otimes E_*^*) = (E_* \otimes E_*) \otimes E_*^* \xrightarrow{\varphi\otimes\mathrm{Id}} L_* \otimes E_*^*.$$

**Definition 2.1.** A *parabolic symplectic bundle* is a pair $(E_*, \varphi)$ of the above form such that $\varphi$ is anti-symmetric, and the homomorphism $\widetilde{\varphi}$ in (2.9) is an isomorphism.

A *parabolic orthogonal bundle* is a pair $(E_*, \varphi)$ of the above form such that $\varphi$ is symmetric, and the homomorphism $\widetilde{\varphi}$ is an isomorphism.

### *2.4. Equivalence with other definitions in the case of rational parabolic weights*

When all the parabolic weights are rational, principal bundles with parabolic structure were defined in [2,9]. We will show that Definition 2.1 coincides with the definition in [2,9] when the parabolic weights are rational.

In this subsection we assume that all the parabolic weights are rational numbers.

We recall that there is a natural correspondence between parabolic vector bundles on $X$ and orbifold vector bundles after we fix a suitable ramified Galois covering of $X$ depending on the common denominator of the parabolic weights [10]. This correspondence takes the dual of a parabolic vector bundle $E_*$ to the usual dual of the orbifold vector bundle corresponding to $E_*$; it takes the tensor product of two parabolic vector bundles to the usual tensor product of the corresponding orbifold vector bundles.

Therefore, if $(E_*, \varphi)$ is a parabolic symplectic or orthogonal vector bundle (see Definition 2.1), then $\varphi$ induces a bilinear form $\varphi'$ on the orbifold vector bundle $E'$ corresponding to the parabolic vector bundle $E_*$. This form $\varphi'$ takes values in the orbifold line bundle $L'$ corresponding to the parabolic line bundle $L_*$. We note that $\varphi'$ is nondegenerate because $\widetilde{\varphi}$ in (2.9) is an isomorphism. Therefore, $(E', \varphi')$ is an orbifold symplectic or orthogonal vector bundle depending on whether $(E_*, \varphi)$ is symplectic or orthogonal. Hence $(E_*, \varphi)$ defines a principal parabolic bundle in the sense of [2,9] (see [2, pp. 350–351, Theorem 4.3], [9, p. 124, Theorem 1.1]). The converse also follows similarly.

### *2.5. Adjoint bundle*

Let $E_*$ be a parabolic vector bundle over $X$; the vector bundle underlying this parabolic vector bundle will be denoted by $E$. Let $U$ be a Zariski open subset of $X$, and let

$$T : E|_U \longrightarrow E|_U$$

be an $\mathcal{O}_U$-linear homomorphism. This homomorphism is called *parabolic* if for every $x_i \in U \bigcap S$,

$$T(F_{i,j}) \subset F_{i,j}$$

for every $j \in [1, a_i]$ (see (2.2)). Let

$$\mathrm{End}^p(E_*) \subset \mathrm{End}(E) := E \otimes E^*$$

be the coherent subsheaf defined by the sheaf of all parabolic endomorphisms of $E$. As mentioned in Section 2.3, $\mathrm{End}^p(E_*)$ is the vector bundle underlying the parabolic vector bundle $E_* \otimes E_*^*$.

Now take a homomorphism $\varphi : E_* \otimes E_* \longrightarrow L_*$ such that $(E_*, \varphi)$ is a symplectic or orthogonal parabolic bundle. Let $L$ be the line bundle underlying the parabolic bundle $L_*$. Since the vector bundle underlying $E_* \otimes E_*$ contains $E \otimes E$ as a subsheaf, the homomorphism $\varphi$ gives a homomorphism

$$\varphi_0 : E \otimes E \longrightarrow L.$$

Let

$$\mathrm{ad}(E_*, \varphi) \subset \mathrm{End}^p(E_*) \tag{2.10}$$

be the subbundle generated by the sheaf of homomorphisms

$$T : E \longrightarrow E$$

lying in $\mathrm{End}^p(E_*)$ such that

$$\varphi_0(T(\alpha) \otimes \beta) + \varphi_0(\alpha \otimes T(\beta)) = 0$$

for all locally defined sections $\alpha$ and $\beta$ of $E$.

The vector bundle $\mathrm{ad}(E_*, \varphi)$ defined in (2.10) will be called the *adjoint* bundle of $(E_*, \varphi)$.

The rank of $\mathrm{ad}(E_*, \varphi)$ is the dimension of the orthogonal or symplectic group corresponding to $\varphi$. Let

$$\mathrm{End}^0(E) \subset \mathrm{End}(E)$$

be the subbundle of corank one defined by the sheaf of endomorphisms of $E$ of trace zero. Since $\varphi|_{X \setminus S}$ is nondegenerate, we have

$$\mathrm{ad}(E_*, \varphi)|_{X \setminus S} \subset \mathrm{End}^0(E)|_{X \setminus S}.$$

Since $X \setminus S$ is Zariski open, this implies that

$$\mathrm{ad}(E_*, \varphi) \subset \mathrm{End}^0(E). \tag{2.11}$$

We will give another description of the subbundle $\mathrm{ad}(E_*, \varphi)$. For any locally defined parabolic homomorphism $T : E_* \longrightarrow E_*$, we have the dual homomorphism

$$T^* : E_*^* \longrightarrow E_*^*.$$

The section $T$ of $\mathrm{End}^p(E_*)$ lies in $\mathrm{ad}(E_*, \varphi)$ if and only if the following diagram is commutative

$$\begin{array}{ccc} E_* & \xrightarrow{\widetilde{\varphi}} & E_*^* \otimes L_* \\ \downarrow T & & \downarrow T^* \otimes \mathrm{Id}_L \\ E_* & \xrightarrow{\widetilde{\varphi}} & E_*^* \otimes L_* \end{array}$$

where $\widetilde{\varphi}$ is the isomorphism in (2.9).

Equip the line bundle $\mathcal{O}_X(S)$ with the trivial parabolic structure (so there is no nonzero parabolic weight). Consider the parabolic vector bundle $\mathrm{ad}(E_*, \varphi) \otimes \mathcal{O}_X(S)$. Let

$$\mathrm{ad}^0(E_*, \varphi) \subset \mathrm{ad}(E_*, \varphi) \otimes \mathcal{O}_X(S) \tag{2.12}$$

be the coherent subsheaf defined by the sheaf of all locally defined sections

$$T : E_* \longrightarrow E_* \otimes \mathcal{O}_X(S)$$

of $\mathrm{ad}(E_*, \varphi) \otimes \mathcal{O}_X(S)$ such that $T(F_{i,j}) \subset F_{i,j+1}$ for all $x_i \in S$ in the domain of $T$ and all $j \in [1, a_i]$ (see (2.2)).

Let $\text{tr} : \text{End}(E) \otimes \text{End}(E) \longrightarrow \mathcal{O}_X$ be the homomorphism defined by $A \otimes B \longmapsto \text{trace}(A \circ B)$. Consider the composition

$$\text{ad}(E_*, \varphi) \otimes \text{ad}^0(E_*, \varphi) \hookrightarrow \text{End}(E) \otimes (\text{End}(E) \otimes \mathcal{O}_X(S)) \stackrel{\text{tr}\otimes\text{Id}}{\longrightarrow} \mathcal{O}_X(S).$$

The image of this composition homomorphism is the subsheaf $\mathcal{O}_X \subset \mathcal{O}_X(S)$, and the above pairing

$$\text{ad}(E_*, \varphi) \otimes \text{ad}^0(E_*, \varphi) \longrightarrow \mathcal{O}_X$$

is nondegenerate. Hence we get an isomorphism of vector bundles

$$\text{ad}^0(E_*, \varphi) \stackrel{\sim}{\longrightarrow} \text{ad}(E_*, \varphi)^*. \tag{2.13}$$

## 3. Connection on parabolic orthogonal and symplectic bundles

### 3.1. Logarithmic connection

Let $\Omega_X$ be the canonical line bundle of $X$. A *logarithmic connection* on a holomorphic vector bundle $W \longrightarrow X$ singular over $S$ is a first order holomorphic differential operator

$$D : W \longrightarrow W \otimes \Omega_X \otimes \mathcal{O}_X(S)$$

satisfying the Leibniz identity which says that $D(fs) = fD(s) + s \otimes df$, where $f$ is a locally defined holomorphic function and $s$ is a locally defined holomorphic section of $W$. See [11] for the details.

Since $S$ is fixed, we will often refrain from referring to $S$. A logarithmic connection on $X$ will always mean that the singular locus of the logarithmic connection is contained in $S$.

For notational convenience, the line bundle $\Omega_X \otimes \mathcal{O}_X(S)$ will be denoted by $\Omega_X(S)$. Take any point $x_i \in S$. Using the Poincaré adjunction formula, the fiber $\Omega_X(S)_{x_i}$ is identified with $\mathbb{C}$. We recall that if $f$ is a function defined on an open neighborhood of $x_i$ such that $f(x_i) = 0$, and $df(x_i) \neq 0$, then the evaluation of the section $(df)/f$ of $\Omega_X(S)$ at $x_i$ is 1. For a logarithmic connection $(V, D)$, consider the composition

$$V \stackrel{D}{\longrightarrow} V \otimes \Omega_X(S) \longrightarrow (V \otimes \Omega_X(S))_{x_i} = V_{x_i}, \tag{3.1}$$

where $V \otimes \Omega_X(S) \longrightarrow (V \otimes \Omega_X(S))_{x_i}$ is the restriction map. This composition is $\mathcal{O}_X$-linear due to the Leibniz identity, hence it defines an endomorphism of the complex vector space $V_{x_i}$. This endomorphism is denoted by

$$\text{Res}(D, x_i) \in \text{End}_{\mathbb{C}}(V_{x_i}),$$

and it is called the *residue* of $D$ at $x_i$; see [11, p. 53].

Let $E_* = (E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ be a parabolic vector bundle. An *algebraic connection* on $E_*$ is a logarithmic connection $D$ on $E$ such that for all $x_i \in S$, and all $j \in [1, a_i]$ (see (2.2) for $a_i$), the following two conditions hold:

$$\text{Res}(D, x_i)(F_{i,j}) \subseteq F_{i,j} \tag{3.2}$$

(this condition implies that $\text{Res}(D, x_i)$ induces an endomorphism of the quotient vector space $F_{i,j}/F_{i,j+1}$), and

$$\text{Res}(D, x_i)|_{F_{i,j}/F_{i,j+1}} = \alpha_{i,j} \cdot \text{Id}_{F_{i,j}/F_{i,j+1}}, \tag{3.3}$$

where $\alpha_{i,j}$ is the parabolic weight in (2.3).

**Lemma 3.1.** *If $E_*$ admits an algebraic connection, then par-deg$(E_*) = 0$.*

**Proof.** Let $W$ be a vector bundle over $X$ equipped with a logarithmic connection $D$ singular over $S$. Then

$$\text{degree}(V) + \sum_{i=1}^{n} \text{trace}(\text{Res}(D, x_i)) = 0$$

[12, p. 16, Theorem 3]. In view of this, the lemma follows immediately from (3.3) and the definition of parabolic degree given in (2.4). □

**Lemma 3.2.** *Let $E_*$ and $F_*$ be parabolic vector bundles equipped with algebraic connections $D_E$ and $D_F$ respectively. Then $D_E$ and $D_F$ together induce an algebraic connection on $E_* \otimes F_*$. Also, $D_E$ induce an algebraic connection on $E_*^*$.*

**Proof.** Let $E$ and $F$ be the vector bundles underlying $E_*$ and $F_*$, respectively. The logarithmic connection $D_E$ on $E$ induces a logarithmic connection on the dual vector bundle $E^*$; this logarithmic connection on $E^*$ will be denoted by $D'_E$. Let $E_0^*$ be the vector bundle underlying the parabolic vector bundle $E_*^*$. This $E_0^*$ is a subsheaf of $E^*$. It is straightforward to check that the logarithmic connection $D'_E$ on $E^*$ produces a logarithmic connection on $E_0^*$, and the resulting logarithmic connection on $E_0^*$ is an algebraic connection on the parabolic vector bundle $E_*^*$.

The two logarithmic connections $D_E$ and $D_F$ together define a logarithmic connection on $E \otimes F$. The line bundle $\mathcal{O}_X(S)$ has a natural logarithmic connection given by the de Rham differential $f \longmapsto \mathrm{d}f$. This logarithmic connection on $\mathcal{O}_X(S)$ and the above logarithmic connection on $E \otimes F$ together define a logarithmic connection on $E \otimes F \otimes \mathcal{O}_X(S)$.

The vector bundle $(E_* \otimes F_*)_0$ underlying the parabolic tensor product $E_* \otimes F_*$ is a subsheaf of $E \otimes F \otimes \mathcal{O}_X(S)$. It is straightforward to check that the above logarithmic connection on $E \otimes F \otimes \mathcal{O}_X(S)$ produces a logarithmic connection on $(E_* \otimes F_*)_0$, and the resulting logarithmic connection on $(E_* \otimes F_*)_0$ is an algebraic connection on the parabolic vector bundle $E_* \otimes F_*$. □

It is known that a parabolic line bundle $E_*$ on $X$ of degree zero admits an algebraic connection. In fact $E_*$ has a unique unitary flat connection [13–15]. We include a simple proof.

**Lemma 3.3.** *Let $E_*$ be a parabolic line bundle with par-deg$(E_*) = 0$. Then $E_*$ admits an algebraic connection.*

**Proof.** For any $x_i \in S$, let $0 \le \lambda_i < 1$ be the parabolic weight of $E_*$ over $x_i$. Let $d$ be the degree of the vector bundle $E$ underlying $E_*$. So,

$$\text{par-deg}(E_*) = d + \sum_{i=1}^{n} \lambda_i = 0. \tag{3.4}$$

An algebraic connection on $E_*$ is a logarithmic connection on $E$ with residue $\lambda_i$ at each point $x_i \in S$.

Fix a divisor $\Delta_E = \sum_{j=1}^{m+d} y_j - \sum_{k=1}^{m} z_k$ such that $E = \mathcal{O}_X(\Delta_E)$. We may, and we will, assume that $x_i, y_j$ and $z_k$ are all distinct points. As mentioned before, $\mathcal{O}_X(\Delta_E)$ has a tautological logarithmic connection $\mathcal{D}_0$ defined by the de Rham differential defined by $f \longmapsto \mathrm{d}f$. This logarithmic connection $\mathcal{D}_0$ is singular over the points $\{y_j\}_{j=1}^{m+d}$ and $\{z_k\}_{j=1}^{m}$, and its residue over each $y_j$ is $-1$ and its residue over each $z_k$ is 1. To prove that $E_*$ admits an algebraic connection, it suffices to produce a section $\omega$ of the line bundle

$$\mathcal{L} := \Omega_X \otimes \mathcal{O}_X(S) \otimes \mathcal{O}_X\left(\sum_{j=1}^{m+d} y_j\right) \otimes \mathcal{O}_X\left(\sum_{k=1}^{m} z_k\right) \tag{3.5}$$

such that the residue of $\omega$ over every $x_i \in S$ is $\lambda_i$, over each $y_j$ is 1 and over each $z_k$ is $-1$. Indeed, the logarithmic connection $\mathcal{D}_0 + \omega$ on $E$, where $\omega$ is a section of the line bundle $\mathcal{L}$ in (3.5) satisfying the above residue conditions, is an algebraic connection on $E_*$.

To construct such a section $\omega$, consider the short exact sequence of coherent sheaves on $X$

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_{S+\sum_{j=1}^{m+d} y_j + \sum_{k=1}^{m} z_k} \longrightarrow 0.$$

Let

$$H^0(X, \mathcal{L}) \longrightarrow \left(\bigoplus_{x_i \in S} \mathcal{L}|_{x_i}\right) \oplus \left(\bigoplus_{j=1}^{m+d} \mathcal{L}|_{y_j}\right) \oplus \left(\bigoplus_{k=1}^{m} \mathcal{L}|_{z_k}\right) \longrightarrow H^1(X, \Omega_X) = \mathbb{C} \tag{3.6}$$

be the corresponding long exact sequence of cohomologies. From (3.6) it follows that $\mathcal{L}$ has a section with the given residues over $S$, $\{y_j\}_{j=1}^{m+d}$ and $\{z_k\}_{j=1}^{m}$ as long as the sum of all the residues is zero. Therefore, from (3.4) we conclude that there is a section $\omega$ such that the residue of $\omega$ over each $x_i \in S$ is $\lambda_i$, over each $y_j$ is 1 and over each $z_k$ is $-1$. This completes the proof of the lemma. □

### *3.2. Definition of an algebraic connection*

Let $E_* = (E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ be a parabolic vector bundle of rank $r$. Let $L_*$ be a parabolic line bundle. Let

$$\varphi : E_* \otimes E_* \longrightarrow L_*$$

be an orthogonal or symplectic parabolic structure. We have

$$\text{par-deg}(L_* \otimes E_*^*) = r \cdot \text{par-deg}(L_*) + \text{par-deg}(E_*^*) = r \cdot \text{par-deg}(L_*) - \text{par-deg}(E_*).$$

Hence from the isomorphism $\widetilde{\varphi}$ in (2.9) it follows that

$$r \cdot \text{par-deg}(L_*) = 2 \cdot \text{par-deg}(E_*). \tag{3.7}$$

Therefore,

$$\text{par-deg}(L_*) = 0 \iff \text{par-deg}(E_*) = 0. \tag{3.8}$$

In this subsection we assume that par-deg$(L_*) = 0$.

Since par-deg$(L_*) = 0$, from Lemma 3.3 we know that $L_*$ has an algebraic connection. Fix an algebraic connection $D_L$ on $L_*$.

As before, let $(E_*, \varphi)$ be a symplectic or orthogonal parabolic bundle. Let $D$ be an algebraic connection on the parabolic vector bundle $E_*$. The algebraic connection $D$ on $E_*$ induces an algebraic connection on $E_*^*$. This induced algebraic connection on $E_*^*$ and the algebraic connection $D_L$ on $L_*$ together produce an algebraic connection on $E_*^* \otimes L_*$ (see Lemma 3.2).

**Definition 3.4.** The algebraic connection on $D$ on $E_*$ is said to be *compatible* with $\varphi$ if the isomorphism $\widetilde{\varphi} : E_* \longrightarrow L_* \otimes E_*^*$ in (2.9) takes the algebraic connection $D$ on $E_*$ to the algebraic connection on $L_* \otimes E_*^*$ constructed above from $D$ and $D_L$.

An *algebraic connection* on $(E_*, \varphi)$ is an algebraic connection on $E_*$ compatible with $\varphi$.

Let $D$ be an algebraic connection on the parabolic vector bundle $E_*$. We will describe a criterion for $D$ to be compatible with $\varphi$.

Let $D'$ be the algebraic connection on $E_* \otimes E_*$ induced by $D$ (see Lemma 3.2). The algebraic connection $D$ is compatible with $\varphi$ if and only if the homomorphism $\varphi$ intertwines $D'$ and $D_L$ (the given algebraic connection on $L_*$).

The algebraic connections $D$ and $D_L$ together produce an algebraic connection on the parabolic tensor product $L_* \otimes E_*^* \otimes E_*^*$. On the other hand, $\varphi$ defines a section of $L_* \otimes E_*^* \otimes E_*^*$. The homomorphism $\varphi$ intertwines $D'$ and $D_L$ if and only if the section of $L_* \otimes E_*^* \otimes E_*^*$ defined by $\varphi$ is flat with respect to the algebraic connection on $L_* \otimes E_*^* \otimes E_*^*$ constructed using $D$ and $D_L$.

If $D_1$ and $D_2$ are algebraic connections on $(E_*, \varphi)$, then

$$D_1 - D_2 \in H^0(X, \mathrm{ad}^0(E_*, \varphi) \otimes \Omega_X),$$

where $\mathrm{ad}^0(E_*, \varphi)$ is the vector bundle in (2.12). Conversely, for any algebraic connection $D$ on $(E_*, \varphi)$, and for any

$$\theta \in H^0(X, \mathrm{ad}^0(E_*, \varphi) \otimes \Omega_X),$$

their sum $D + \theta$ is also an algebraic connection on $(E_*, \varphi)$. Therefore, the following holds:

**Lemma 3.5.** *The space of all algebraic connections on $(E_*, \varphi)$ is an affine space for the vector space $H^0(X, \mathrm{ad}^0(E_*, \varphi) \otimes \Omega_X)$.*

### 3.3. Generalized logarithmic connection

Let $U \subset X$ be a nonempty Zariski open subset. The intersection $U \bigcap S$ will be denoted by $S_U$. The canonical line bundle of $U$ will be denoted by $\Omega_U$. Fix an algebraic function $w$ on $U$. Let $V$ be an algebraic vector bundle on $U$.

A *generalized logarithmic connection on $V$ with weight $w$* is an algebraic differential operator

$$D : V \longrightarrow V \otimes \Omega_U(S_U) := V \otimes \Omega_U \otimes \mathcal{O}_U(S_U)$$

satisfying the identity

$$D(fs) = fD(s) + w \cdot s \otimes \mathrm{d}f, \tag{3.9}$$

where $f$ is a locally defined algebraic function, and $s$ is a locally defined algebraic section of $V$.

The identity in (3.9) implies that the order of the differential operator $D$ is at most one. The order of $D$ is zero, meaning $D$ is $\mathcal{O}_U$-linear, if and only if $w = 0$. We also note that $D$ is a logarithmic connection over $U$ if $w = 1$.

A *generalized logarithmic connection* on $V \longrightarrow U$ is a pair $(w, D)$, where $w$ is an algebraic function on $U$, and $D$ is a generalized logarithmic connection on $V$ with weight $w$.

Given any generalized logarithmic connection $(w, D)$ on $V \longrightarrow U$, for any point $x_i \in S_U$, consider the composition

$$V \xrightarrow{D} V \otimes \Omega_U \otimes \mathcal{O}_U(S_U) =: V \otimes \Omega_U(S_U) \longrightarrow (V \otimes \Omega_U(S_U))_{x_i} = V_{x_i}$$

as in (3.1). It defines an endomorphism

$$\mathrm{Res}(D, x_i) \in \mathrm{End}_{\mathbb{C}}(V_{x_i});$$

this endomorphism will be called the *residue* of $D$ at $x_i$.

Let $E_* = (E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ be a parabolic vector bundle. Let $U \subset X$ be a nonempty Zariski open subset. A *generalized algebraic connection* on $E_*|_U$ is generalized logarithmic connection $(w, D)$ on $E|_U$ satisfying the following condition: for any $x_i \in U \bigcap S$,

$$\mathrm{Res}(D, x_i)(F_{i,j}) \subseteq F_{i,j} \quad \text{and} \quad \mathrm{Res}(D, x_i)|_{F_{i,j}/F_{i,j+1}} = w(x_i) \cdot \alpha_{i,j} \cdot \mathrm{Id}_{F_{i,j}/F_{i,j+1}}$$

for all $j \in [1, a_i]$; the first condition ensures that $\mathrm{Res}(D, x_i)$ induces an endomorphism of $F_{i,j}/F_{i,j+1}$.

### 3.4. Connections and Atiyah exact sequence

As in Section 3.2, we assume that $\text{par-deg}(L_*) = 0$. We also fix an algebraic connection $D_L$ on $L_*$.

Let $E_* = (E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ be a parabolic vector bundle, and let $(E_*, \varphi)$ be an orthogonal or symplectic parabolic bundle. Let $U \subset X$ be a nonempty Zariski open subset, and let $w$ be a function on $U$. Note that $w \cdot D_L$ is a generalized logarithmic connection on $L|_U$ of weight $w$.

If $V$ (respectively, $W$) is a vector bundle on $U$ equipped with a generalized logarithmic connection $D_V$ (respectively, $D_W$) of weight $w$, then $D_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes D_W$ is a generalized logarithmic connection on $V \otimes W$ of weight $w$. Also, $D_V$ induces a generalized logarithmic connection $D_{V^*}$ on the dual vector bundle $V^*$ of weight $w$. This $D_{V^*}$ is uniquely defined by the following identity:

$$t(D_V(s)) + (D_V^*(t))(s) = w \cdot d(t(s)),$$

where $s$ and $t$ are locally defined sections of $V$ and $V^*$ respectively.

Consequently, for any generalized algebraic connection $D$ of weight $w$ on the parabolic vector bundle $E_*|_U$, we have a generalized algebraic connection $D^*$ on $E_*^*|_U$ of weight $w$. Also, $D^*$ and $w \cdot D_L$ together produce a generalized algebraic connection on $E_*^* \otimes L_*$ of weight $w$.

A generalized algebraic connection on $D$ of weight $w$ on the parabolic vector bundle $E_*|_U$ is said to be *compatible* with $\varphi$ if the isomorphism $\widetilde{\varphi}$ in (2.9) takes $D$ to the generalized algebraic connection on $E_*^* \otimes L_*$ constructed using $D$ and $w \cdot D_L$.

A *generalized algebraic connection* on $(E_*, \varphi)$ over $U$ is a generalized algebraic connection on the parabolic vector bundle $E_*$ compatible with $\varphi$.

Let

$$\mathcal{D}(E_*, \varphi) \tag{3.10}$$

be the sheaf on $X$ defined by the generalized algebraic connections on $(E_*, \varphi)$.

We will show that $\mathcal{D}(E_*, \varphi)$ defines an algebraic vector bundle over $X$. For that purpose, first note that if $(w_1, D_1)$ and $(w_2, D_2)$ are generalized algebraic connections on $(E_*, \varphi)$ over $U$, then $(w_1 + w_2, D_1 + D_2)$ is a generalized algebraic connection on $(E_*, \varphi)$ over $U$. Also, if $w$ is a function on $U$, then $(w \cdot w_1, w \cdot D_1)$ is a generalized algebraic connection on $(E_*, \varphi)$ over $U$. Consequently, $\mathcal{D}(E_*, \varphi)$ defines a coherent sheaf on $X$. Clearly, the $\mathcal{O}_X$-module $\mathcal{D}(E_*, \varphi)$ is torsionfree, and its rank coincides with $1 + \mathrm{rank}(\mathrm{ad}(E_*, \varphi))$ (see (2.10) for $\mathrm{ad}(E_*, \varphi)$). Hence $\mathcal{D}(E_*, \varphi)$ is a vector bundle over $X$ of rank $1 + \mathrm{rank}(\mathrm{ad}(E_*, \varphi))$.

Let

$$\eta : \mathcal{D}(E_*, \varphi) \longrightarrow \mathcal{O}_X \tag{3.11}$$

be the homomorphism defined by $(w, D) \longmapsto w$. From the $\mathcal{O}_X$-module structure of $\mathcal{D}(E_*, \varphi)$ described above it follows immediately that $\eta$ is $\mathcal{O}_X$-linear. Also, $\eta$ is surjective.

The sheaf of generalized algebraic connections of weight zero on $(E_*, \varphi)$ coincides with the sheaf of sections of the vector bundle $\mathrm{ad}^0(E_*, \varphi) \otimes \Omega_X$, where $\mathrm{ad}^0(E_*, \varphi)$ is constructed in (2.12). Hence we have an inclusion

$$\mathrm{ad}^0(E_*, \varphi) \otimes \Omega_X \hookrightarrow \mathcal{D}(E_*, \varphi).$$

Using this inclusion we get a short exact sequence of vector bundles on $X$

$$0 \longrightarrow \mathrm{ad}^0(E_*, \varphi) \otimes \Omega_X \longrightarrow \mathcal{D}(E_*) \xrightarrow{\eta} \mathcal{O}_X \longrightarrow 0, \tag{3.12}$$

where $\eta$ is the homomorphism in (3.11).

The short exact sequence in (3.12) is a twisted form of the Atiyah exact sequence. More precisely, if the parabolic structure on $E_*$ is trivial, then (3.12) tensored with $TX$ coincides with the usual Atiyah exact sequence for the corresponding orthogonal or symplectic bundle. We recall that an algebraic connection on a principal bundle is an algebraic splitting of the Atiyah exact sequence associated to the principal bundle [4]. Also, note that splittings of a given short exact sequence are in bijective correspondence with the splittings of the short exact sequence obtained by tensoring the given exact sequence with some line bundle.

Recall the definition of an algebraic connection on $(E_*, \varphi)$ (see Definition 3.4). If

$$\sigma : \mathcal{O}_X \longrightarrow \mathcal{D}(E_*, \varphi)$$

is a homomorphism such that $\eta \circ \sigma = \mathrm{Id}_{\mathcal{O}_X}$, where $\eta$ is the homomorphism in (3.11), then the section $\sigma(1)$ of $\mathcal{D}(E_*, \varphi)$ is an algebraic connection on $(E_*, \varphi)$; here 1 is the section of $\mathcal{O}_X$ given by the constant function 1. Conversely, if $D$ is an algebraic connection on $(E_*, \varphi)$, then there is a unique homomorphism

$$\sigma : \mathcal{O}_X \longrightarrow \mathcal{D}(E_*, \varphi)$$

such that $\sigma(1) = D$. In other words, algebraic connections on $(E_*, \varphi)$ are the splitting of the short exact sequence in (3.12).

## 4. Criterion for an algebraic connection

We assume that $\text{par-deg}(L_*) = 0$. Fix an algebraic connection $D_L$ on the parabolic line bundle $L_*$. Let

$$(E_*, \varphi) = ((E, \{F_{i,j}\}, \{\alpha_{i,j}\}), \varphi)$$

be an orthogonal or symplectic parabolic bundle. In this section, we will give a criterion for $(E_*, \varphi)$ to admit an algebraic connection.

*4.1. The case of symplectic bundles*

First assume that $\varphi$ is alternating. Let $2r$ be the rank of $E$.

Take any parabolic vector bundle $V_*$. Using the natural pairing of $V_*$ with the parabolic dual $V_*^*$, the parabolic vector bundle $V_* \oplus (V_*^* \otimes L_*)$ is equipped with a symplectic form with values in $L_*$. Let

$$\varphi^a_{V_*} : (V_* \oplus (V_*^* \otimes L_*)) \otimes (V_* \oplus (V_*^* \otimes L_*)) \longrightarrow L_*$$

be this symplectic form on $V_* \oplus (V_*^* \otimes L_*)$.

**Theorem 4.1.** *The parabolic symplectic bundle* $(E_*, \varphi)$ *admits an algebraic connection if and only if the following holds: for every parabolic vector bundle* $V_*$ *satisfying the condition that there is a symplectic parabolic vector bundle* $(W_*, \phi)$ *such that*

$$(E_*, \varphi) = ((V_* \oplus (V_*^* \otimes L_*)) \oplus W_*, \varphi^a_{V_*} \oplus \phi),$$

*we have*

$$\text{par-deg}(V_*) = 0.$$

**Proof.** First assume that $(E_*, \varphi)$ admits an algebraic connection. Take a parabolic vector bundle $V_*$, and a symplectic parabolic vector bundle $(W_*, \phi)$, such that

$$(E_*, \varphi) = ((V_* \oplus (V_*^* \otimes L_*)) \oplus W_*, \varphi^a_{V_*} \oplus \phi).$$

Fix an isomorphism $\tau : (E_*, \varphi) \longrightarrow ((V_* \oplus (V_*^* \otimes L_*)) \oplus W_*, \varphi^a_{V_*} \oplus \phi)$. Let

$$i_V : V_* \longrightarrow (V_* \oplus (V_*^* \otimes L_*)) \oplus W_* \quad \text{and} \quad j_V := (V_* \oplus (V_*^* \otimes L_*)) \oplus W_* \longrightarrow V_* \tag{4.1}$$

be the injection and projection respectively constructed using $\tau$.

Let $D$ be an algebraic connection on $(E_*, \varphi)$. Consider the composition

$$V_* \xrightarrow{i_V} (V_* \oplus (V_*^* \otimes L_*)) \oplus W \xrightarrow{D} ((V_* \oplus (V_*^* \otimes L_*)) \oplus W) \otimes \Omega_X \otimes \mathcal{O}_X(S)$$
$$\xrightarrow{j_V \otimes \mathrm{Id}_{\Omega_X \otimes \mathcal{O}_X(S)}} V_* \otimes \Omega_X \otimes \mathcal{O}_X(S).$$

It is an algebraic connection on the parabolic vector bundle $V_*$. Now from Lemma 3.1 we conclude that par-deg$(V_*) = 0$.

To prove the converse, we will first show that it is enough to prove the converse under the assumption that $(E_*, \varphi)$ is irreducible, meaning it does not decompose into a direct sum of symplectic parabolic vector bundles of positive rank.

We can write $(E_*, \varphi)$ as a direct sum

$$(E_*, \varphi) = \bigoplus_{i=1}^{n} (E^i_*, \varphi^i),$$

where each $(E^i_*, \varphi^i)$ is irreducible. If the condition in the theorem holds for $(E_*, \varphi)$, then it holds for each $(E^i_*, \varphi^i)$. If each symplectic parabolic vector bundle $(E^i_*, \varphi^i)$ has an algebraic connection $D^i$, then $\bigoplus_{i=1}^n D^i$ is an algebraic connection on $(E_*, \varphi)$. Therefore, it is enough to prove the converse under the assumption that $(E_*, \varphi)$ is irreducible.

We assume that $(E_*, \varphi)$ is irreducible. Assume that the condition in the theorem holds for $(E_*, \varphi)$.

We will show that the short exact sequence in (3.12) splits; recall that any splitting of (3.12) is an algebraic connection on $(E_*, \varphi)$.

Using the Serre duality, the obstruction to the splitting of (3.12) is a cohomology class

$$c \in H^1(X, \mathrm{ad}^0(E_*, \varphi) \otimes \Omega_X) = H^0(X, \mathrm{ad}(E_*, \varphi))^* \tag{4.2}$$

(see (2.13)). We will investigate the functional $c$ of $H^0(X, \mathrm{ad}(E_*, \varphi))$.

Take any $A \in H^0(X, \mathrm{ad}(E_*, \varphi))$. So, $A$ is an endomorphism of $E$ compatible with $\varphi$ which preserves the quasi-parabolic flags for $E_*$. Consider the characteristic polynomial of $A(x)$, $x \in X$. Since there are no nonconstant algebraic functions on $X$, the coefficients of the characteristic polynomial of $A(x)$ are independent of $x$. Hence the eigenvalues of $A(x)$, along with their multiplicities, are independent of $x$. Since $(E_*, \varphi)$ is irreducible, there is exactly one eigenvalue (otherwise the generalized eigenspace decomposition of $E$ contradicts irreducibility). On the other hand, $A$ is of trace zero (see (2.11)). Hence $A$ does not have any nonzero eigenvalue.

Since $A$ does not have any nonzero eigenvalue, we conclude that the endomorphism $A$ is nilpotent. Now from the argument in the proof of Proposition 18(ii) in [4, p. 202] it follows that the functional $c$ in (4.2) satisfies the identity $c(A) = 0$. Hence $c = 0$. Therefore, $(E_*, \varphi)$ admits an algebraic connection. □

### 4.2. The case of orthogonal bundles

We now consider the case where $(E_*, \varphi)$ is an orthogonal parabolic vector bundle.

Take any parabolic vector bundle $V_*$. Using the natural pairing of $V_*$ with the parabolic dual $V_*^*$, the parabolic vector bundle $V_* \oplus (V_*^* \otimes L_*)$ is equipped with an orthogonal form with values in $L_*$. Let

$$\varphi^s_{V_*} : (V_* \oplus (V_*^* \otimes L_*)) \otimes (V_* \oplus (V_*^* \otimes L_*)) \longrightarrow L_*$$

be this orthogonal form on $V_* \oplus (V_*^* \otimes L_*)$.

**Theorem 4.2.** *The orthogonal parabolic bundle $(E_*, \varphi)$ admits an algebraic connection if and only if the following holds: for every parabolic vector bundle $V_*$ satisfying the condition that there is an orthogonal parabolic vector bundle $(W_*, \phi)$ such that*

$$(E_*, \varphi) = ((V_* \oplus (V_*^* \otimes L_*)) \oplus W_*, \varphi^s_{V_*} \oplus \phi),$$

*we have*

$$\textit{par-deg}(V_*) = 0.$$

The proof of Theorem 4.2 is identical to the proof of Theorem 4.1.

In the absence of any parabolic structure, Theorems 4.1 and 4.2 coincide with the main theorem of [16] (Theorem 4.1) for symplectic and orthogonal bundles respectively.

## 5. Semistable and polystable parabolic bundles

### 5.1. Semistability of tensor product

Let $E_* = (E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ be a parabolic vector bundle over $X$. Any subbundle $F$ of $E$ is equipped with an induced parabolic structure which is obtained by restricting the quasi-parabolic filtrations and the parabolic weights of $E$ to $F$. Let $F_*$ be the parabolic vector bundle obtained this way.

The parabolic vector bundle $E_*$ is called *stable* (respectively, *semistable*) if for every subbundle $F \subset E$ with $0 < \text{rank}(F) < \text{rank}(E)$, the inequality

$$\mu_{\text{par}}(F_*) < \mu_{\text{par}}(E_*) \quad (\text{respectively, } \mu_{\text{par}}(F_*) \leq \mu_{\text{par}}(E_*)) \tag{5.1}$$

holds (see [1], [5, p. 69, Définition 6]).

The parabolic vector bundle $E_*$ is called *polystable* if is semistable, and isomorphic to a direct sum of stable parabolic vector bundles.

Fix a complete Hermitian metric $g_X$ on $X \setminus S$; it is Kähler because $\dim_{\mathbb{C}} X = 1$. The notion of an Einstein–Hermitian metric on vector bundles over $X$ extends to a notion of an Einstein–Hermitian metric on parabolic vector bundles (see [14, p. 492, Definition 5.7], [13] for the details).

The following is a basic theorem (see [14, p. 497, Theorem 6.4], [13, p. 718, Theorem]):

**Theorem 5.1.** *A parabolic vector bundle $E_*$ admits an Einstein–Hermitian connection if and only if $E_*$ is polystable, and the Einstein–Hermitian connection on a polystable parabolic bundle is unique.*

A $C^\infty$ connection on $E_*$ is a $C^\infty$ splitting of the short exact sequence in (3.12). An Einstein–Hermitian connection is not algebraic unless it is flat.

**Lemma 5.2.** *If $E_*$ and $V_*$ are parabolic polystable vector bundles, then the parabolic tensor product $E_* \otimes V_*$ is also polystable.*

**Proof.** Both $E_*$ and $V_*$ admit Einstein–Hermitian connection by Theorem 5.1. The connection on $E_* \otimes V_*$ induced by Einstein–Hermitian connections on $E_*$ and $V_*$ is again Einstein–Hermitian. Hence $E_* \otimes V_*$ is polystable. □

Any semistable parabolic vector bundle admits a filtration of subbundles such that each successive quotient is polystable of same parabolic slope. Therefore, Lemma 5.2 has the following corollary:

**Corollary 5.3.** *If $E_*$ and $V_*$ are parabolic semistable vector bundles, then the parabolic tensor product $E_* \otimes V_*$ is also semistable.*

### 5.2. Semistable and stable orthogonal and symplectic bundles

Let $V$ be a finite dimensional complex vector space equipped with an orthogonal form $B$. Let $\text{SO}(V) \subset \text{SL}(V)$ be the subgroup consisting of all automorphisms that preserve $B$. Let

$$\text{GO}(V) \subset \text{GL}(V)$$

be the subgroup consisting of all automorphisms $T$ satisfying the condition that there is a constant $c \in \mathbb{C}^*$ such that

$$B(T(v), T(w)) = c \cdot B(v, w)$$

for all $v, w \in V$. So $\mathrm{GO}(V)$ fits in a short exact sequence

$$e \longrightarrow \mathrm{SO}(V) \longrightarrow \mathrm{GO}(V) \longrightarrow \mathbb{C}^* \longrightarrow e.$$

A linear subspace $V_0 \subset V$ is called *isotropic* if $B(v, w) = 0$ for all $v, w \in V_0$. The subgroup of $\mathrm{GO}(V)$ that preserves a fixed nonzero isotropic subspace $V_0$ is a maximal parabolic subgroup of $\mathrm{GO}(V)$. In fact all maximal parabolic subgroups of $\mathrm{GO}(V)$ arise this way. Let $V_0$ be a nonzero isotropic subspace of $V$. Let

$$P \subset \mathrm{GO}(V)$$

be the corresponding maximal parabolic subgroup. We will describe a Levi subgroup of $P$. Consider the orthogonal subspace $V_0^\perp \subset V$ for $V_0$. Since $V_0$ is isotropic, we have $V_0 \subset V_0^\perp$. Fix a complement $W_0 \subset V_0^\perp$ of the subspace $V_0$. The restriction of $B$ to $W_0$ is nondegenerate. Fix an isotropic subspace $V_1 \subset V$ such that $V_1$ is a complement of $V_0^\perp$. The subgroup of $P$ consisting of all automorphisms preserving both $W_0$ and $V_1$ is a Levi subgroup of $P$. All Levi subgroups of $P$ are of this form for some choices of $W_0$ and $V_1$.

If $B'$ is a symplectic form on $V$, then define

$$\mathrm{Gp}(V) \subset \mathrm{GL}(V)$$

to be the subgroup consisting of all automorphisms $T$ satisfying the condition that there is $c \in \mathbb{C}^*$ such that

$$B'(T(v), T(w)) = c \cdot B'(v, w)$$

for all $v, w \in V$. As before, a linear subspace $V_0 \subset V$ is called *isotropic* if $B(v, w) = 0$ for all $v, w \in V_0$. Maximal parabolic subgroups of $\mathrm{Gp}(V)$, and the Levi subgroups of maximal parabolic subgroups, have exactly identical description as those for $\mathrm{GO}(V)$.

Let $E$ be a holomorphic vector bundle on $X$, and let $L$ be a holomorphic line bundle on $X$. Let

$$\varphi_0 : E \otimes E \longrightarrow L$$

be a nondegenerate bilinear form which is either symmetric or anti-symmetric. So $(E, \varphi_0)$ defines a principal $\mathrm{GO}(V)$-bundle or a principal $\mathrm{Gp}(V)$-bundle on $X$ depending on whether $\varphi_0$ is symmetric or anti-symmetric, where $V$ is as before with $\dim V = \mathrm{rank}(E)$.

A holomorphic subbundle $F \subset E$ is called *isotropic* if $\varphi_0 \left(F \bigotimes F\right) = 0$.

Combining the above descriptions of maximal parabolic and Levi subgroups with the definition of a (semi)stable principal bundle (see [17, page 129, Definition 1.1] and [17, page 131, Lemma 2.1]), we get the following:

The principal bundle defined by $(E, \varphi_0)$ is stable (respectively, semistable) if and only if for all nonzero isotropic subbundles $F \subset E$,

$$\frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} < \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)} \quad \left(\text{respectively, } \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \leq \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}\right).$$

The principal bundle defined by $(E, \varphi_0)$ is polystable if and only if either $(E, \varphi_0)$ is stable, or there is a polystable vector bundle $W$ with

$$\frac{\mathrm{degree}(W)}{\mathrm{rank}(W)} = \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)},$$

and an orthogonal or symplectic stable parabolic vector bundle $(F, \phi)$ (depending on whether $(E, \varphi_0)$ is orthogonal or symplectic), with $\phi$ taking values in the same line bundle $L$ as for $\varphi_0$, such that $(E, \varphi_0)$ is isomorphic to the direct sum

$$((W \oplus (W^* \otimes L)) \oplus F, \varphi_W \oplus \phi),$$

where $\varphi_W$ is the obvious orthogonal or symplectic structure on $W \oplus (W^* \otimes L)$ constructed using the natural paring of $W$ with its dual $W^*$.

### *5.3. Semistable and stable orthogonal and symplectic parabolic bundles*

Let $(E_*, \varphi) = ((E, \{F_{i,j}\}, \{\alpha_{i,j}\}), \varphi)$ be an orthogonal or symplectic parabolic bundle. To clarify, we no longer assume that the parabolic degree of $L_*$ is zero.

A subbundle $F$ of $E$ is called *isotropic* if the restriction of $\varphi$ to $F \otimes F$ vanishes identically.

The bundle $(E_*, \varphi)$ is called *stable* (respectively, *semistable*) if

$$\mu_{\mathrm{par}}(F_*) < \mu_{\mathrm{par}}(E_*) \quad (\text{respectively, } \mu_{\mathrm{par}}(F_*) \leq \mu_{\mathrm{par}}(E_*))$$

for all nonzero isotropic subbundles $F$ of $E$ with $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$.

The bundle $(E_*, \varphi)$ is called *polystable* if either $(E_*, \varphi)$ is stable, or there is a parabolic polystable vector bundle $V_*$ with $\mu_{\text{par}}(V_*) = \mu_{\text{par}}(E_*)$, and an orthogonal or symplectic stable parabolic vector bundle $(F_*, \phi)$ (depending on whether $(E_*, \varphi)$ is orthogonal or symplectic) such that $(E_*, \varphi)$ is isomorphic to the direct sum

$$((V_* \oplus (V_*^* \otimes L_*)) \oplus F_*, \varphi_{V_*} \oplus \phi),$$

where $\varphi_{V_*}$ is the obvious orthogonal or symplectic structure on $V_* \oplus (V_*^* \otimes L_*)$ constructed using the natural paring of $V_*$ with its parabolic dual $V_*^*$ (so $\varphi_{V_*}$ is either $\varphi^a_{V_*}$ in Theorem 4.1, or $\varphi^s_{V_*}$ in Theorem 4.2); the parabolic vector bundle $F_*$ is allowed to be zero.

It is easy to see that if $(E_*, \varphi)$ is polystable, then $(E_*, \varphi)$ is semistable.

In the above definition, the condition that $\mu_{\text{par}}(V_*) = \mu_{\text{par}}(E_*)$ implies that

$$\mu_{\text{par}}(V_*^* \otimes L_*) = \mu_{\text{par}}(E_*)$$

because $\mu_{\text{par}}(V_* \oplus (V_*^* \otimes L_*)) = \mu_{\text{par}}(E_*)$ (see (3.7)). If $F_*$ in the above definition is nonzero, then $\mu_{\text{par}}(F_*) = \mu_{\text{par}}(E_*)$ from (3.7).

### 5.4. Harder–Narasimhan filtration

Let $(E_*, \varphi) = ((E, \{F_{i,j}\}, \{\alpha_{i,j}\}), \varphi)$ be an orthogonal or symplectic parabolic bundle. Assume that the parabolic vector bundle $E_*$ is not semistable. Then it has a unique Harder–Narasimhan filtration

$$V_*^1 \subset V_*^2 \subset \cdots \subset V_*^{n-1} \subset V_*^n = E_* \tag{5.2}$$

(see [5, p. 70, Théorème 8]). Consider the filtration of $E_*^* \otimes L_*$

$$E_*^* \otimes L^* \twoheadrightarrow (V_*^{n-1})^* \otimes L_* \twoheadrightarrow \cdots \twoheadrightarrow (V_*^1)^* \otimes L_*, \tag{5.3}$$

where $V_*^i$, $1 \le i \le n$, are as in (5.2) (the kernels of the projections from $E_*^* \otimes L^*$ produce the filtration in (5.3)). From the definition of a Harder–Narasimhan filtration it follows immediately that (5.3) is the Harder–Narasimhan filtration of $E_*^* \otimes L_*$.

Any isomorphism between two parabolic vector bundles preserves the Harder–Narasimhan filtrations, because the Harder–Narasimhan filtration is unique. Therefore, the isomorphism $\widetilde{\varphi}$ in (2.9) takes the filtration in (5.2) to the filtration in (5.3). Therefore, we have the following proposition:

**Proposition 5.4.** *For any $i \in [1, n-1]$, the image $\widetilde{\varphi}(V_*^i)$ coincides with the kernel of the projection $E_*^* \otimes L^* \twoheadrightarrow (V_*^{n-i})^* \otimes L_*$ in (5.3).*

Since $V_*^1 \subset V_*^{n-1}$, Proposition 5.4 has the following corollary:

**Corollary 5.5.** *The subbundle $V_*^1 \subset E_*$ in (5.2) is isotropic.*

**Proposition 5.6.** *Let $(E_*, \varphi) = ((E, \{F_{i,j}\}, \{\alpha_{i,j}\}), \varphi)$ be an orthogonal or symplectic parabolic bundle. Then $(E_*, \varphi)$ is semistable if and only if the parabolic vector bundle $E_*$ is semistable.*

**Proof.** If the parabolic vector bundle $E_*$ is semistable, then obviously $(E_*, \varphi)$ is semistable. To prove the converse, assume that the parabolic vector bundle $E_*$ is not semistable. Let (5.2) be the Harder–Narasimhan filtration of $E_*$. Since $\mu_{\text{par}}(V_*^1) > \mu_{\text{par}}(E_*)$, and $V_*^1 \subset E_*$ is isotropic (see Corollary 5.5), we conclude that $V_*^1$ violates the semistability criterion for $(E_*, \varphi)$. Therefore, $(E_*, \varphi)$ is not semistable. □

### 5.5. The socle filtration

Let $E_* = (E, \{F_{i,j}\}, \{\alpha_{i,j}\})$ be a semistable parabolic vector bundle. If $V_*$ and $W_*$ are polystable nonzero subbundles of $E_*$ with

$$\mu_{\text{par}}(V_*) = \mu_{\text{par}}(W_*) = \mu_{\text{par}}(E_*),$$

then the parabolic subbundle $F_* \subset E_*$ generated by $V_*$ and $W_*$ is also polystable with $\mu_{\text{par}}(F_*) = \mu_{\text{par}}(E_*)$; the proof of it is identical to that of [18, p. 23, Lemma 1.5.5].

Therefore, there is a unique maximal polystable parabolic subbundle $E'_* \subset E_*$ such that $\mu_{\text{par}}(E'_*) = \mu_{\text{par}}(E_*)$. This parabolic subbundle $E'_*$ is called the *socle* of $E_*$.

For the socle $E'_* \subset E_*$, if $E_*/E'_* \neq 0$, then the parabolic vector bundle $E_*/E'_*$ is semistable, and $\mu_{\text{par}}(E_*/E'_*) = \mu_{\text{par}}(E_*)$. We may consider the socle of $E_*/E'_*$. Hence there is a unique filtration of parabolic subbundles

$$0 = E_*^0 \subset E_*^1 \subset E_*^2 \subset \cdots \subset E_*^{m-1} \subset E_*^m = E_* \tag{5.4}$$

such that for each $i \in [1, m-1]$, the quotient $E_*^i/E_*^{i-1}$ is the socle of $E_*/E_*^{i-1}$.

Since $E_*$ is semistable, the parabolic vector bundle $E_*^* \otimes L_*$ is semistable. The filtration

$$E_*^* \otimes L^* \twoheadrightarrow (E_*^{m-1})^* \otimes L_* \twoheadrightarrow \cdots \twoheadrightarrow (E_*^1)^* \otimes L_* \tag{5.5}$$

obtained from (5.4) clearly coincides with the socle filtration of $E_*^* \otimes L^*$.

Let $\varphi$ be a bilinear form $E_*$ such that $(E_*, \varphi)$ is an orthogonal or symplectic parabolic vector bundle. From the uniqueness of the socle filtration it follows immediately that the isomorphism $\widetilde{\varphi}$ in (2.9) takes the filtration in (5.4) to the filtration in (5.5).

**Proposition 5.7.** *Let $(E_*, \varphi)$ be a polystable orthogonal or symplectic parabolic bundle. Then the parabolic vector bundle $E_*$ is polystable.*

**Proof.** Since $(E_*, \varphi)$ is polystable, it is semistable. Hence $E_*$ is semistable by Proposition 5.6. Let (5.4) be the socle filtration of $E_*$. It was noted above that the isomorphism $\widetilde{\varphi}$ in (2.9) takes the filtration in (5.4) to the filtration in (5.5).

We assume that $E_*$ is not polystable. Therefore, we have $\text{rank}(E_*^1) < \text{rank}(E_*)$, where $E_*^1$ is the socle of $E_*$ in (5.4).

Since $\widetilde{\varphi}$ takes $E_*^1$ isomorphically to the kernel of the projection $E_*^* \otimes L^* \longrightarrow (E_*^{m-1})^* \otimes L_*$, it follows immediately that

$$\varphi(E_*^1, E_*^{m-1}) = 0.$$

We now conclude that $E_*^1$ is isotropic because $E_*^1 \subset E_*^{m-1}$. Since $(E_*, \varphi)$ is polystable, and $E_*^1$ is an isotropic subbundle with $\mu_{\text{par}}(E_*^1) = \mu_{\text{par}}(E_*)$, we conclude that there is an orthogonal or symplectic parabolic vector bundle $(F_*, \phi)$ (depending on whether $(E_*, \varphi)$ is orthogonal or symplectic) such that $(E_*, \varphi)$ is isomorphic to the direct sum

$$((E_*^1 \oplus ((E_*^1)^* \otimes L_*)) \oplus F_*, \varphi_{E_*^1} \oplus \phi),$$

where $\varphi_{E_*^1}$ is the obvious orthogonal or symplectic structure on $E_*^1 \oplus ((E_*^1)^* \otimes L_*)$ constructed using the natural paring of $E_*^1$ with its parabolic dual $(E_*^1)^*$ (so $\varphi_{E_*^1}$ is either $\varphi^a_{E_*^1}$ in Theorem 4.1, or $\varphi^s_{E_*^1}$ in Theorem 4.2).

Note that $\mu_{\text{par}}(E_*^1) = \mu_{\text{par}}((E_*^1)^* \otimes L_*) = \mu_{\text{par}}(F_*)$. Since the parabolic vector bundle $E_*$ is isomorphic to $E_*^1 \oplus ((E_*^1)^* \otimes L_*) \oplus F_*$, we have a contradiction to the fact that $E_*^1$ is the maximal polystable parabolic subbundle of $E_*$ with parabolic slope $\mu_{\text{par}}(E_*)$. Therefore, we conclude that $E_*$ is polystable. □

## 6. Einstein–Hermitian connection on orthogonal or symplectic parabolic bundle

As in Section 5.1, fix a complete Hermitian metric $g_X$ on $X \setminus S$. Let $\nabla_L$ be the Einstein–Hermitian connection on $L_*$ given by Theorem 5.1 (we do not assume that $\text{par-deg}(L_*) = 0$).

Let $(E_*, \varphi) = ((E, \{F_{i,j}\}, \{\alpha_{i,j}\}), \varphi)$ be an orthogonal or symplectic parabolic bundle; the form $\varphi$ takes values in $L_*$. If $D$ is an Einstein–Hermitian connection on $E_*$, then the connection $D^*$ on $E_*^*$ induced by $D$ is also Einstein–Hermitian. The Einstein–Hermitian connection $D^*$ on $E_*^*$ and the Einstein–Hermitian connection $\nabla_L$ on $L_*$ together define an Einstein–Hermitian connection on $E_*^* \otimes L_*$.

An *Einstein–Hermitian connection* on $(E_*, \varphi)$ is an Einstein–Hermitian connection $D$ on $E_*$ such that the isomorphism $\widetilde{\varphi}$ in (2.9) takes $D$ to the connection on $E_*^* \otimes L_*$ constructed using $D^*$ and $\nabla_L$.

**Theorem 6.1.** *Let $(E_*, \varphi)$ be an orthogonal or symplectic parabolic bundle. Then $(E_*, \varphi)$ admits an Einstein–Hermitian connection if and only if $(E_*, \varphi)$ is polystable.*

**Proof.** If $(E_*, \varphi)$ admits an Einstein–Hermitian connection, then the proof of the "only if" part of Theorem 5.1 gives that $(E_*, \varphi)$ is polystable. (It should be clarified that the nontrivial part of Theorem 5.1 is that a polystable parabolic bundle admits an Einstein–Hermitian connection.)

To prove the converse, assume that $(E_*, \varphi)$ is polystable. Then the parabolic vector bundle $E_*$ is polystable by Proposition 5.7. Let $D$ be the unique Einstein–Hermitian connection on $E_*$ (Theorem 5.1). The connection on $E_*^* \otimes L_*$ constructed using $D^*$ and $\nabla_L$ is Einstein–Hermitian. On the other hand, the connection on $E_*^* \otimes L_*$ given by $D$ using the isomorphism $\widetilde{\varphi}$ in (2.9) is also Einstein–Hermitian. Now from the uniqueness of the Einstein–Hermitian connection on a polystable parabolic vector bundle it follows immediately that the above two connections on $E_*^* \otimes L_*$ coincide. □

We have the following converse of Proposition 5.7.

**Corollary 6.2.** *Let $(E_*, \varphi)$ be an orthogonal or symplectic parabolic bundle such that $E_*$ is polystable. Then $(E_*, \varphi)$ is polystable.*

**Proof.** Let $D$ be the Einstein–Hermitian connection on $E_*$ given by Theorem 5.1. We saw in the proof of Theorem 6.1 that $D$ is an Einstein–Hermitian connection on $(E_*, \varphi)$. Therefore, $(E_*, \varphi)$ is polystable by Theorem 6.1. □

It should be mentioned that if $(E_*, \varphi)$ is stable, then $E_*$ is not stable in general. To construct such examples, take stable bundles $(E_*, \varphi)$ and $(F_*, \phi)$ with both orthogonal or both symplectic, and both $\varphi$ and $\phi$ taking values in a fixed line bundle $L_*$. Then $(E_* \oplus F_*, \varphi \oplus \phi)$ is also stable, but $E_* \oplus F_*$ is not stable.

## References

[1] C.S. Seshadri, Moduli of vector bundles on curves with parabolic structures, Bull. Amer. Math. Soc. 83 (1977) 124–126.
[2] V. Balaji, I. Biswas, D.S. Nagaraj, Principal bundles over projective manifolds with parabolic structure over a divisor, Tohoku Math. J. 53 (2001) 337–367.
[3] A. Weil, Generalisation des fonctions abeliennes, J. Math. Pures Appl. 17 (1938) 47–87.
[4] M.F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957) 181–207.
[5] C.S. Seshadri, Fibrés vectoriels sur les courbes algébriques, in: Astérisque, vol. 96, Société Math. de Fr., 1982.
[6] M. Maruyama, K. Yokogawa, Moduli of parabolic stable sheaves, Math. Ann. 293 (1992) 77–99.
[7] I. Biswas, Parabolic ample bundles, Math. Ann. 307 (1997) 511–529.
[8] K. Yokogawa, Infinitesimal deformation of parabolic Higgs sheaves, Int. J. Math. 6 (1995) 125–148.
[9] V. Balaji, I. Biswas, D.S. Nagaraj, Ramified *G*-bundles as parabolic bundles, J. Ramanujan Math. Soc. 18 (2003) 123–138.
[10] I. Biswas, Parabolic bundles as orbifold bundles, Duke Math. J. 88 (1997) 305–325.
[11] P. Deligne, Equations Différentielles à Points Singuliers Réguliers, in: Lecture Notes in Math., vol. 163, Springer-Verlag, Berlin, 1970.
[12] M. Ohtsuki, A residue formula for Chern classes associated with logarithmic connections, Tokyo J. Math. 5 (1982) 13–21.
[13] C.T. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc. 3 (1990) 713–770.
[14] J.A. Poritz, Parabolic vector bundles and Hermitian–Yang–Mills connections over a Riemann surface, Int. J. Math. 4 (1993) 467–501.
[15] O. Biquard, Fibrés paraboliques stables et connexions singulières plates, Bull. Soc. Math. France 119 (1991) 231–257.
[16] H. Azad, I. Biswas, On holomorphic principal bundles over a compact Riemann surface admitting a flat connection, Math. Ann. 322 (2002) 333–346.
[17] A. Ramanathan, Stable principal bundles on a compact Riemann surface, Math. Ann. 213 (1975) 129–152.
[18] D. Huybrechts, M. Lehn, The geometry of moduli spaces of sheaves, in: Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.